Chapter 2

Induction

2.1 Introducing induction

Suppose there is an infinite line of people, numbered $1, 2, 3, \ldots$, and every person has been instructed as follows: "If something is whispered in your ear, go ahead and whisper the same thing to the person in front of you (the one with the greater number)". Now, what will happen if we whisper a secret to person 1? 1 will tell it to 2, 2 will tell it to 3, 3 will tell it to 4, and ... everybody is going to learn the secret! Similarly, suppose we align an infinite number of dominoes, such that if some domino falls, the next one in line falls as well. What happens when we knock down the first domino? That's right, they all fall. This intuition is formalized in the principle of mathematical induction:

Induction Principle: Given a set A of positive integers, suppose the following hold:

- $1 \in A$.
- If $k \in A$ then $k + 1 \in A$.

Then *all* positive integers belong to A. (That is, $A = \mathbb{N}^+$.)

Here are two simple proofs that use the induction principle:

Theorem 2.1.1. *Every positive integer is either even or odd.*

Proof. By definition, we are required to prove that for every $n \in \mathbb{N}^+$, there exists some $l \in \mathbb{N}$, such that either $n = 2l$ or $n = 2l + 1$. The proof proceeds by induction. The claim holds for $n = 1$, since $1 = 2 \cdot 0 + 1$. Suppose the claim holds for $n = k$. That is, there exists $l \in \mathbb{N}$, such that $k = 2l$ or $k = 2l + 1$. We prove that the claim holds for $n = k + 1$. Indeed, if $k = 2l$ then $k + 1 = 2l + 1$, and if $k = 2l + 1$ then $k + 1 = 2(l + 1)$. Thus the claim holds for $n = k + 1$ and the proof by induction is complete. $\square$

Theorem 2.1.2. *Every positive integer power of 3 is odd.*

Proof. By definition, we are required to prove that for every $n \in \mathbb{N}^+$, it holds that $3^n = 2l + 1$, for some $l \in \mathbb{N}$. The proof proceeds by induction. For $n = 1$, we have $3 = 2 \cdot 1 + 1$, so the claim holds. Suppose the claim holds for k, so $3^k = 2l + 1$, for some $l \in \mathbb{N}$. Then

$$3^{k+1} = 3 \cdot 3^k = 3(2l + 1) = 2(3l + 1) + 1,$$

and the claim also holds for $k + 1$. The proof by induction is complete. $\square$

Proof tip: If you don't know how to get a proof started, look to the definitions, and state formally and precisely what it is that you need to prove. It might not be obvious how to prove that "Every positive integer power of 3 is odd", but a bit easier to proceed with proving that "for every $n \in \mathbb{N}^+$, it holds that $3^n = 2l + 1$, for some $l \in \mathbb{N}$." If you need to prove an implication (that is, a claim of the form "if ... then ..."), then formally state all the assumptions as well as what you need to prove that they imply. Comparing the two might lead to some insight.

Proof technique: Induction. The induction principle is often used when we are trying to prove that some claim holds for all positive integers. As the above two proofs illustrate, when we use induction we do not need to explicitly refer to the set A from the statement of the induction principle. Generally, this set is the set of numbers for which the claim that we are trying to prove holds. In the first proof, it was the set of numbers n that are either even or odd. In the second proof, it was the set of numbers n for which 3^n is odd. Suppose we want to show that some claim holds for all positive integers. Here is a general template for proving this by induction:

(a) State the method of proof. For example, "The proof proceeds by induction."

(b) Prove the "induction basis". That is, prove that the number 1 satisfies the claim. (This step is often easy, but is crucially important, and should never be omitted!)

(c) Assume the "induction hypothesis". That is, state the assumption that the claim holds for some positive integer k.

(d) Prove, using the induction hypothesis, that the claim holds for $k+1$. The proof should consist of a chain of clear statements, each logically following from the previous ones combined with our shared knowledge base. The final statement in the chain should state that the claim holds for $k + 1$.

(e) Conclude the proof. For example, "This completes the proof by induction."

Theorem 2.1.3. *For every positive integer n,*

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof. The proof proceeds by induction. For $n = 1$, we have $1 = \frac{1 \cdot 2}{2}$ and the claim holds. Assume $1 + 2 + \cdots + k = k(k + 1)/2$. Then

$$1+2+\cdots+k+(k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2},$$

which proves the claim for $k + 1$ and completes the proof by induction. $\square$

Sigma and Pi notations. Just as the $\bigcup$ symbol can be used to compactly express the union of many sets, the $\sum$ symbol can be used to express summations. For example,

$$1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \sum_{1 \le i \le n} i = \sum_{i \in \{x \,:\, 1 \le x \le n\}} i.$$

You should not assume just because $\sum$ appears that there is an actual summation, or that there are any summands at all. For example, when $n = 1$, $\sum_{i=1}^{n} i = 1$, and when $n \le 0$, $\sum_{i=1}^{n} i = 0$!

Similarly, products can be expressed using the $\prod$ symbol, as in

$$2^0 \cdot 2^1 \cdot 2^2 \cdot \ldots \cdot 2^n = \prod_{i=0}^{n} 2^i.$$

One thing to be aware of is that the empty product is defined to equal 1, so

$$\prod_{i=3}^{1} i \quad = \quad \prod_{\substack{i \,\in\, \{2,4,10,14\} \\ i \text{ is odd}}} i \quad = \quad 1.$$

A single $\sum$ or $\prod$ symbol can also be used to describe the sum or product over more than one variable. For example,

$$\sum_{1 \le i,j \le n} (i + j) = \sum_{i=1}^{n} \sum_{j=1}^{n} (i + j).$$

2.2 Strong induction

Suppose that a property P holds for $n = 1$, and the following is true: If P holds for all integers between 1 and k, then it also holds for $k + 1$. Under these assumptions, P holds for all positive integers. This is the principle of strong induction. It differs from regular induction in that we can assume something stronger to derive the same conclusion. Namely, we can assume not only that P holds for k, but that in fact P holds for all positive integers up to k. We state the strong induction principle more formally, and then demonstrate its usefulness.

Strong Induction Principle: Given a set A of positive integers, suppose the following hold:

- $1 \in A$.
- If $\{1, 2, \ldots, k\} \subseteq A$ then $k + 1 \in A$.

Then all positive integers belong to A.

Definition. An integer $p > 1$ is said to be *prime* if the only positive divisors of p are 1 and p itself.

Theorem 2.2.1. *Every positive integer greater than* 1 *can be expressed as a product of primes.*

Proof. The proof proceeds by strong induction. Since 2 is a prime, the claim holds for 2. (Note how the induction basis in this case is 2, not 1, since we are proving a claim concerning all integers equal to or greater than 2.) Now assume the claim holds for all integers between 2 and k. If $k+1$ is a prime then the claim trivially holds. Otherwise it has a positive divisor a other than 1 and $k + 1$ itself. Thus, $k + 1 = a \cdot b$, with $2 \leq a, b \leq k$. Both a and b can be expressed as products of primes by the induction hypothesis. Their product can therefore also be thus expressed. This completes the proof by strong induction. □

The versatility of induction. We have seen in the proof of Theorem 2.2.1 that if we want to prove a statement concerning all positive integers equal to or greater than 2, we can use induction (or strong induction) with 2 as the base case. This holds for any positive integer in the place of 2. In fact, induction is an extremely versatile technique. For example, if we want to prove a property of all even positive integers, we can use 2 as the base case, and then prove that if the property holds for k, it will also hold for $k + 2$. Generally we will just assume that such variations are ok, there is no need to state a separate induction principle for each of these cases.

Fairly subtle variations of induction are often used. For example, if we can prove that a statement holds for 1 and 2, and that if it holds for k it will also hold for $k + 2$, we can safely conclude that the statement holds for all the positive integers. However, don't get carried away with variations that are simply incorrect, like using 1 as a base case, proving that if a statement holds for k then it also holds for $k + 2$, and then claiming its validity for all positive integers.

2.3 Why is the induction principle true?

Some of you might be surprised by the title question. Isn't it obvious? I mean, you know, the dominoes are aligned, you knock one down, they all fall. End of story. Right? Not quite.

"Common sense" often misleads us. You probably noticed this in daily life, and you're going to notice it a whole lot if you get into mathematics. Think of optical illusions: we see, very clearly, what isn't really there. Our mind plays tricks on us too, just like our eyes sometimes do. So in mathematics, we are after proving everything. To be mathematically correct, every statement has to logically follow from previously known ones. So how do we prove the induction principle?

The answer lies in the previous paragraph. We said that every statement has to logically follow from other statements that we have proven previously. But this cannot go on forever, do you see? We have to start from some statements that we *assume* to be true. Such statements are called axioms. For example, why is it true that for any two natural numbers a, b, c, it holds that $a+(b+c) = (a+b)+c$? Because we assume it to be so, in order to build up the rest of mathematics from this and a small number of other such axioms.

This is also what we do with the induction principle: We accept it as an axiom. And if we accept the induction principle, strong induction can be proved from it, as you'll discover in the homework.